Some elements of nonlinear elasticity of growing materials

Michael H. Köpf

January 8, 2015

A very good introduction to nonlinear elasticity can be found in the book by Fu and Ogden [FO01]. A necessary starter to dive into elasticity theory is certainly the classical textbook by Landau and Lifshitz [LL70].

In the following, the component notation $c_i = A_{ij} b_j$ with Einstein summation over repeated indices is preferred over the notation $\mathbf{c} = \mathbf{A} \cdot \mathbf{b}$, because the latter has to be translated to the former for every actual calculation, anyway.

1 Basic definitions

The elastic deformation can quantified by comparison of the deformed state $\mathcal{B}$ to a reference state $\mathcal{A}$. We denote the coordinates in the reference frame $\mathcal{A}$ by $x_i^{(\mathcal{A})}$ and use $X_i^{(\mathcal{B})}$ for the coordinates in the current state $\mathcal{B}$[1]. Further, we introduce a mapping $\mathbf{F}$ that relates the coordinates of identical material points in A to the corresponding coordinates in B (see Fig. 1). In order to simplify our notation, we introduce the shorthands $\partial_i^{(\mathcal{A})} := \partial/\partial x_i^{(\mathcal{A})}$ and $\partial_i^{(\mathcal{B})} := \partial/\partial x_i^{(\mathcal{B})}$ for the partial derivatives with respect to the coordinates in both frames. We define the *deformation gradient tensor* $\mathbf{F}$ by

$$F_{ij} = \partial_j^{(\mathcal{A})} x_i^{(\mathcal{B})} \tag{1}$$

[1]Note, that the majority of the literature uses lower case coordinate names x_i, in the deformed frame and capital coordinate names X_i in the reference frame.

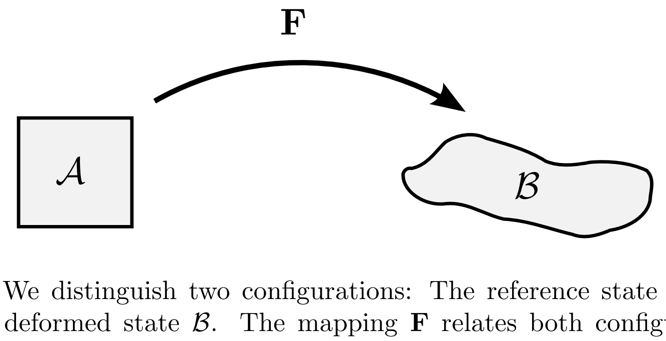

Figure 1: We distinguish two configurations: The reference state $\mathcal{A}$ and the elastically deformed state $\mathcal{B}$. The mapping $\mathbf{F}$ relates both configurations to each other.

and introduce the symbol $J_F = |\mathbf{F}|$ for its determinant. The deformation of the solid is quantified by the *strain* ω defined as [2]

$$\omega_{ij} = F_{ki} F_{kj} - \delta_{ij}. \tag{2}$$

Nanson's formula In order to relate the oriented area element $\mathrm{d}A_i^{(\mathcal{A})}$ in frame $\mathcal{A}$ to $\mathrm{d}A_i^{(\mathcal{B})}$ in frame $\mathcal{B}$, we consider infinitesimal volume elements in both frames $\mathcal{A}$ and $\mathcal{B}$ and use $\mathrm{d}x_i^{(\mathcal{B})} = F_{ij}\mathrm{d}x_j^{(\mathcal{A})}$:

$$\mathrm{d}V^{(\mathcal{B})} = J_F dV^{(\mathcal{A})} = J_F \mathrm{d}A_i^{(\mathcal{A})} \mathrm{d}x_i^{(\mathcal{A})} \tag{3}$$

and

$$\mathrm{d}V^{(\mathcal{B})} = \mathrm{d}A_i^{(\mathcal{B})} \mathrm{d}x_i^{(\mathcal{B})} = \mathrm{d}A_i^{(\mathcal{B})} F_{ij} \mathrm{d}x_j^{(\mathcal{A})}. \tag{4}$$

From these two formulas we conclude

$$\begin{aligned}
&J_F \mathrm{d}A_i^{(\mathcal{A})} = \mathrm{d}A_k^{(\mathcal{B})} F_{ki} \\
\Leftrightarrow &J_F \left(F^{-1}\right)_{il} \mathrm{d}A_i^{(\mathcal{A})} = \mathrm{d}A_l^{(\mathcal{B})} \\
\Leftrightarrow &J_F \left(F^{-1}\right)_{il} n_i^{(\mathcal{A})} \mathrm{d}A^{(\mathcal{A})} = n_i^{(\mathcal{B})} \mathrm{d}A^{(\mathcal{B})}.
\end{aligned} \tag{5}$$

Analogously, we find

$$n_i^{(\mathcal{A})} \mathrm{d}A^{(\mathcal{A})} = J_F^{-1} F_{ki} n_k^{(\mathcal{B})} \mathrm{d}A^{(\mathcal{B})}. \tag{6}$$

Helpful relations A few non-obvious identities can be found using Nanson's formula and very simple geometric arguments. Consider an arbitrary closed surface $\delta\omega$ of some volume ω in the reference frame $\mathcal{A}$.

$$0 = \int_{\partial\omega} \mathrm{d}A^{(\mathcal{A})} n_i^{(\mathcal{A})} = \int_{\partial\Omega} \mathrm{d}A^{(\mathcal{B})} J_F^{-1} F_{ki} n_k^{(\mathcal{B})} \tag{7}$$

Since we can choose the surface $\partial\omega$ arbitrarily, we find

$$\partial_k^{(\mathcal{B})} \left(J_F^{-1} F_{ki}\right) = 0. \tag{8}$$

In other words: in the sum $\partial_k^{(\mathcal{B})}$ and $J_F^{-1} F_{ki}$ commute. Analogously, we can show that

$$\partial_k^{(\mathcal{A})} \left(J_F \left(F^{-1}\right)_{ki}\right) = 0. \tag{9}$$

Three different stress tensors The surface force per unit area on a vector area element $\mathrm{d}\mathbf{A}^{(\mathcal{B})}$ in the current frame $\mathcal{B}$ is expressed by the *Cauchy stress* Σ_{ij}. Using relation (5) we can translate this to a force per unit area on a vector area element $\mathrm{d}\mathbf{A}^{(\mathcal{A})}$ in the reference frame $\mathcal{A}$.

$$\Sigma_{ij} n_j^{(\mathcal{B})} = \underbrace{J_F \left(F^{-1}\right)_{kj} \Sigma_{ij}}_{=: \sigma_{ik}} n_k^{(\mathcal{A})} \mathrm{d}A^{(\mathcal{A})} = \sigma_{ij} n_j^{(\mathcal{A})} \mathrm{d}A^{(\mathcal{A})}, \tag{10}$$

where we have defined the *first Piola-Kirchhoff stress* $\boldsymbol{\sigma}$. The inverse relation is

$$\Sigma_{ij} = J_F^{-1} F_{jk} \sigma_{ik}. \tag{11}$$

[2] In the notation of Landau and Lifshitz [LL70] $\omega_{ij} = 2u_{ij}$.

Note, that unlike the Cauchy stress $\mathbf{\Sigma}$, the first Piola-Kirchhoff stress $\boldsymbol{\sigma}$ is not symmetric.

Both $\mathbf{\Sigma}\mathrm{d}\mathbf{A}^{(\mathcal{B})}$ and $\boldsymbol{\sigma}\mathrm{d}\mathbf{A}^{(\mathcal{A})}$ are force vectors in the current frame $\mathcal{B}$. Using $\mathbf{F}$, we can map it to the reference configuration $\mathcal{A}$ and obtain

$$\begin{aligned}\left(F^{-1}\right)_{ij}\Sigma_{jk}n_k^{(\mathcal{B})}\mathrm{d}A^{(\mathcal{B})} =& \underbrace{J_F\left(F^{-1}\right)_{ij}\left(F^{-1}\right)_{lk}\Sigma_{kj}}_{=:\, s_{il}}n_l^{(\mathcal{A})}\mathrm{d}A^{(\mathcal{A})}\\ =& \underbrace{\left(F^{-1}\right)_{ij}\sigma_{jk}}_{=:\, s_{ik}}n_k^{(\mathcal{A})}\mathrm{d}A^{(\mathcal{A})}\\ =& s_{ij}n_j^{(\mathcal{A})}\mathrm{d}A^{(\mathcal{A})}\end{aligned} \tag{12}$$

where we defined the *second Piola-Kirchhoff stress* $\mathbf{s}$. The inverse relations are

$$\Sigma_{ij} = J_F^{-1}F_{il}F_{jk}s_{kl} \quad \text{and} \quad \sigma_{ij} = F_{ik}s_{kj}. \tag{13}$$

Note, that $\partial_j^{(\mathcal{B})}\Sigma_{ij}$ on the one hand and $\partial_j^{(\mathcal{A})}\sigma_{ij}$ and $\partial_j s_{ij}$ on the other hand are force *densities* with respect to different volumina and therefore differ by a factor of J_F. This can be easily seen by noting that

$$F_i^{(\mathrm{tot})} = \int_\omega \mathrm{d}V^{(\mathcal{B})}\partial_j^{(\mathcal{A})}\sigma_{ij} \tag{14}$$

and

$$F_i^{(\mathrm{tot})} = \int_\Omega \mathrm{d}V^{(\mathcal{B})}\partial_j^{(\mathcal{A})}\Sigma_{ij} = \int_\omega \mathrm{d}V^{(\mathcal{B})}J_F\partial_j^{(\mathcal{B})}\sigma_{ij}. \tag{15}$$

Relation of the Piola-Kirchhoff stresses to the elastic free energy In Hookean elasticity, where, the Cauchy stress is the derivative of the elastic free energy density per unit volume $\mathcal{L}_{\mathrm{el}}$ with respect to the linear strain $\omega_{ij}^{(\mathrm{lin})} = u_{i,j} + u_{j,i}$:

$$\Sigma_{ij} = 2\frac{\partial\mathcal{L}_{\mathrm{el}}}{\partial\omega_{ij}^{(\mathrm{lin})}}. \tag{16}$$

In nonlinear elasticity, the second Piola-Kirchhoff stress takes the role of the Cauchy stress:

$$s_{ij} = 2\frac{\partial\mathcal{L}_{\mathrm{el}}}{\partial\omega_{ij}}. \tag{17}$$

The first Piola-Kirchhoff stress is given by the derivative of $\partial\mathcal{L}_{\mathrm{el}}$ with respect to the deformation gradient tensor F_{ij}:

$$\begin{aligned}\sigma_{ij} &= \frac{\partial\mathcal{L}_{\mathrm{el}}}{\partial F_{ij}} = \frac{\partial\mathcal{L}_{\mathrm{el}}}{\partial\omega_{kl}}\frac{\partial\omega_{kl}}{\partial F_{ij}} = \frac{1}{2}s_{kl}\left(F_{ik}\delta_{lj} + F_{il}\delta_{kj}\right) = \frac{1}{2}\left(s_{kj} + s_{jk}\right)F_{ik}\\ &= F_{ik}s_{kj}.\end{aligned} \tag{18}$$

In the last step, we have used the symmetry of the second Piola-Kirchhoff stress $\mathbf{s}$.

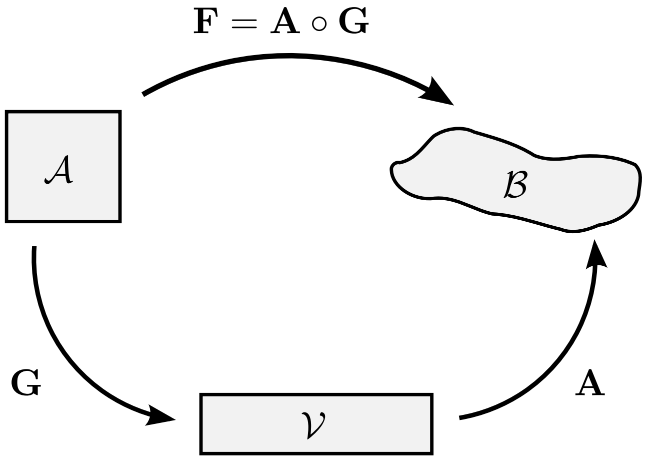

Figure 2: If we have growth and elasticity, we distinguish three configurations: The reference state $\mathcal{A}$, the virtual configuration of a stress-free grown state $\mathcal{V}$, and the actual grown and elastically deformed state $\mathcal{B}$. The mappings $\mathbf{G}$, $\mathbf{A}$, and $\mathbf{F}$ relate the three configurations to each other.

2 Elasticity and growth

When the considered material is growing, for example because it is a biological tissue consisting of proliferating cells, one has to seperate the deformation due to growth from elastic deformation. This is basically the distinction between growth of a spring (that might be caused by uniform thermal expansion) versus extension of the spring by external forces. To make this distinction quantitative, seperate the deformation gradient tensor $\mathbf{F}$ into a product of the two tensors $\mathbf{G}$ and $\mathbf{A}$ as proposed by Rodriguez et al. [RHM94] yielding

$$F_{ij} = A_{ik} G_{kj}, \tag{19}$$

where the *growth tensor* $\mathbf{G}$ describes the deformation due to growth and $\mathbf{A}$ describes the elastic part of the deformation[3]. We can then distinguish not only between the reference state $\mathcal{A}$ and the current deformed state $\mathcal{B}$, but also define the *virtual* state $\mathcal{V}$, that describes the grown but otherwise undeformed material (see Fig. 2). Analogously to the case without growth, we use the shorthands $\partial_i^{(\mathcal{A})} := \partial/\partial x_i^{(\mathcal{A})}$, $\partial_i^{(\mathcal{B})} := \partial/\partial x_i^{(\mathcal{B})}$, and $\partial_i^{(\mathcal{V})} := \partial/\partial x_i^{(\mathcal{V})}$ for the partial derivatives with respect to the different coordinate frames and define $J_F = |\mathbf{F}|$, $J_G = |\mathbf{G}|$, and $J_A = |\mathbf{A}|$. Elastic deformation is deformation from $\mathcal{V}$ to $\mathcal{B}$, only. Therefore, we can for the moment forget about the frame $\mathcal{A}$ and consider $\mathcal{V}$ the reference state. We can then define the virtual Piola-Kirchhoff stress $\sigma^{(\mathcal{V})}$ that translates the force per unit area in the frame $\mathcal{B}$ to a force per unit area in the frame $\mathcal{V}$ as

$$\sigma^{(\mathcal{V})} = \frac{\partial \mathcal{L}_{\mathrm{el}}}{\partial A_{ij}}. \tag{20}$$

[3]The possibility of this sepeartion and its implications are briefly discussed in [Cow04]

Since the physical force is the same, no matter which frame we choose, the following two equalities have to hold:

$$\Sigma_{ij} n_j^{(\mathcal{B})} \mathrm{d}A^{(\mathcal{B})} = \sigma_{ij} n_j^{(\mathcal{A})} \mathrm{d}A^{(\mathcal{A})} = \sigma_{ij}^{(\mathcal{V})} n_j^{(\mathcal{V})} \mathrm{d}A^{(\mathcal{V})}. \tag{21}$$

Using the transformation of the spatial derivative[4] $\partial_j^{(\mathcal{V})} = (G^{-1})_{kj} \partial_k^{(\mathcal{A})}$ one can use Nanson's formula (5) applied to $\mathcal{V}$, $\mathcal{B}$, and $\mathbf{G}$ to find

$$n_i^{(\mathcal{V})} \mathrm{d}A^{(\mathcal{V})} = J_G \left(G^{-1}\right)_{ij} n_j^{(\mathcal{A})} \mathrm{d}A^{(\mathcal{A})}. \tag{22}$$

When we insert this result into eq. (21), we obtain

$$\underbrace{J_G \sigma_{ij}^{(\mathcal{V})} \left(G^{-1}\right)_{kj}}_{=\sigma_{ik}} n_k^{(\mathcal{A})} \mathrm{d}A^{(\mathcal{A})} = \sigma_{ij} n_j^{(\mathcal{A})} \mathrm{d}A^{(\mathcal{A})}, \tag{23}$$

and have thus established the relation between the Piola-Kirchhoff stresses σ_{ij} and $\sigma_{ij}^{(\mathcal{V})}$.

References

[Cow04] Stephen C. Cowin. Tissue growth and remodeling. *Annu. Rev. Biomed. Eng.*, 6(1):77–107, July 2004.

[FO01] Y. B. Fu and R. W. Ogden. *Nonlinear Elasticity: Theory and Applications.* Cambridge University Press, Cambridge, 2001.

[LL70] L. D. Landau and E. M. Lifshitz. *Theory of Elasticity.* Pergamon Press, New York, 2nd edition, 1970.

[RHM94] Edward K. Rodriguez, Anne Hoger, and Andrew D. McCulloch. Stress-dependent finite growth in soft elastic tissues. *Journal of Biomechanics*, 27(4):455–467, April 1994.

[4] Analogously, we have $\partial_i^{(\mathcal{B})} = (A^{-1})_{ji} \partial_j^{(\mathcal{V})}$.